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REPRESENTATION AND DETECTION OF MULTIPLE-EPOCH SIGNAL

By
Tzay Y. Young

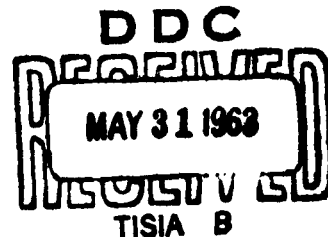
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FOREWORD

This report was prepared by Carlyle Barton Laboratory, The Johns Hopkins University, Baltimore, Maryland on Air Force Contract No. AF 30(602)-2597 under Project No. 4505 of Task No. 450501. The work was administered by the Electronic Warfare Laboratory, Rome Air Development Center. Mr. Haywood E. Webb was the project engineer.

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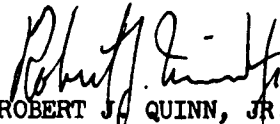
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ABSTRACT

A multiple-epoch signal consists of several signals which for one reason or another, overlap one another. The purpose of this report is to design a procedure for the detection of the individual epochs of the overlapping signals and to represent them properly. The multiple-epoch signal may be corrupted with random Gaussian noise of zero mean. The individual signals are assumed to be representable by a set of exponential functions with acceptable error, and the exponents of this set of exponentials are assumed known. Any two adjacent epochs are assumed to be separated at least T_0 seconds apart.

For the signal uncorrupted with noise a criterion based on the error energy is described which is of theoretical interest. This criterion is useful for the detection of first epoch only. To detect the second epoch, the first signal is subtracted from the original signal by means of the 'complementary' operator concept. For a noisy signal, a likelihood ratio criterion is proposed. Our preliminary experimental results carried out on a digital computer justify the theoretical study.

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I. INTRODUCTION

A multiple-epoch signal consists of several signals which for one reason or another, arrive at the receiver (or measuring apparatus) by different amounts of time delay. The time delays of the individual signals are unknown, and their differences are relatively small so that the signals overlap. The individual signals are of similar nature, but may have different waveshapes. A multiple-epoch signal may be an electro-magnetic signal like radar signal, or biological signal such as speech or electrocardiogram, or signals of other origin. It may be corrupted by random noise. Our problem is then to find the individual epochs of the individual signals in a multiple-epoch signal, and to represent them properly. In other words, our purpose is to separate multiple-epoch signal into overlapping signals, which may be described individually.

To show how we proceed to solve this difficult problem, we would like first to make clear the following assumptions:

1) The individual signals that constitute the multiple-epoch signal are essentially exponential. Or more clearly, these individual signals may themselves be approximated in the least-square sense with acceptable error by a set of exponential functions. The sinusoidal signals are only particular cases of exponential signals. Since the exponential functions serving as component functions for the signal approximation are completed¹, the least-square error in the approximation of any signal may be reduced to as small as we wish by using sufficient number of exponential functions. Therefore the assumption of signals being essentially exponential does not lose the theoretical generality. Yet it does impose some practical limitation to the waveform of the individual signals.

2) A physical signal $v(t)$ starts from $t = 0$ and extends into future. As pointed out by Huggins², any physical operation can operate only upon the past. This nature is inherent in the analysis of signals. The representation and separation of signals cannot be achieved until all the information concerning the multiple epoch signal is available. A very simple and practical solution to this difficulty is to reverse the signal in time. This can be very easily realized on a digital computer or using magnetic tape in the analogue case. Let

$$f(t) = v(-t) \quad , \quad (1.1)$$

then the signal $f(t)$ extends in the interval $-\infty < t \leq 0$. The signal $f(t)$ will be represented (or approximated) by growing exponentials (as versus damped exponentials). In this report, whenever we talk about $f(t)$, we assume they are signals of this nature.

3) The exponents of the exponential functions that constitute the overlapping signals are assumed known a priori. I would like to emphasize here that this assumption is completely unnecessary except for the considerable simplification of presentation. Suppose the multiple-epoch signal $f(t)$ consists of n overlapping signals $f_i(t)$, one may write

$$f(t) = \sum_{i=1}^n f_i(t), \quad -\infty < t \leq 0 \quad , \quad (1.2)$$

where the overlapping signals can be written as

$$\begin{aligned} f_i(t) &\cong \sum_{k=1}^m \{ \xi_{ki} e^{(a_k + j\beta_k)(t-t_i)} + \xi_{ki}^* e^{(a_k - j\beta_k)(t-t_i)} \}, \\ &\quad (-\infty < t \leq t_i) \quad , \\ &= 0 \quad , \quad (t_i < t) \quad . \quad (1.3) \end{aligned}$$

α_k, β_k are positive real numbers and ξ_{ki}, ξ_{ki}^* are complex conjugate coefficients. t_i are individual epochs such that

$$t_1 < t_2 < \dots < t_n = 0 \quad (1.4)$$

It should be noted that ξ_{ki} and ξ_{ki}^* may be zero so that each individual signal may consist of different exponential functions. Our assumption says that α_k and β_k are known while the coefficients ξ_{ki}, ξ_{ki}^* are unknown. In terms of signal space concept, each individual signal may be regarded as a signal vector in the signal space. Then, our assumption means simply that we know the smallest subspace which contains all individual signal vectors, yet the individual signal vectors per se are unknown. In the case that the individual signals are not exactly representable by exponentials, we assume that we know the $2m$ -dimensional exponential subspace that contains the maximum average projection of individual signal vectors.

4) The signal may or may not be corrupted with noise. For the noisy case, it is assumed that the random noise is gaussian and has zero mean. The variance of this gaussian noise is unimportant for our problem. We shall first treat the multiple-epoch signal without noise and then extend the procedure to the noisy case.

5) As we mentioned at the beginning, a multi-epoch signal is formed by several signals whose epochs are different yet close enough to cause overlapping. We now assume that for any two adjacent epochs,

$$t_{i+1} - t_i \geq T_0 \quad (1.5)$$

where T_0 is a predetermined quantity. This is important since we shall utilize the information between t_i and $t_i + T_0$ to separate the signal $f_i(t)$ from the rest. The separation is achieved by means

of the so-called 'complementary' operator concept³. Thus if we have a way to detect the first epoch t_1 , we may utilize the information in the time interval $(t_1, t_1 + T_0)$ to find $f_1(t)$ and subtract it from $f(t)$. The detection of second epoch t_2 is then reduced to the same problem as detecting t_1 . Similarly, once we detect the epoch t_i , we may subtract $f_i(t)$ from the signal $f_i(t) + f_{i+1}(t) + \dots + f_n(t)$ so as to make possible the detection of t_{i+1} . This is indeed the main strategy we propose for the solution of multiple-epoch problems. The procedure should become clear as we proceed to discuss in detail the method.

In this last paragraph of the section, we would like to say a few words about the reason why the third assumption is indeed unnecessary. For a single-epoch signal which is exactly representable by say, $2m$, exponential functions, there exist in literature a Prony's method and its various modified versions to find these exponents provided that the value of m is known⁴. Furthermore this knowledge of m does not impose any limitation at all since there is a method to test the value of m . One may first assign a value M , then if $M > m$, a certain determinant will become singular, and on the other hand if $M \leq m$, this determinant will not be singular. Thus one may take the largest M which does not cause the determinant to become singular as our natural choice.

The Prony's method does not give us the optimum exponents in the least square sense if our signal is to be approximated by $2m$ exponential functions. Recently, McDonough⁴ has shown an iterative method which converges to the least-square optimum exponents. Since the tail of the multiple-epoch signal consists of all the exponentials that constitute the individual signals, we may apply the Prony's method or McDonough's method to the signal $f(t)$ in the

time interval $(-\infty, t_0)$ where $t_0 \leq t_1$. In case the signal is corrupted with relatively large noise, a low-pass filter is required to filter out the high frequency noise before we apply the Prony's or McDonough's method. However, it does not mean that the problem is reduced to the case of noise-free signal. Since the low-pass filtration destroys the epoch information, we shall deal with the original noisy signal in epoch detection.

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II. SIGNAL ANALYSIS USING ORTHONORMAL EXPONENTIALS

The ordinary exponential functions are not orthogonal, i. e., they are correlated with each other. If one uses these functions directly for signal representation, the result will be extremely sensitive to slight numerical errors. The orthonormality of two functions in time domain is defined as

$$\int_0^{\infty} \phi_i(t) \phi_j(t) dt = \delta_{ij}, \quad (2.1)$$

where δ_{ij} is the Kronecker delta,

$$\begin{aligned} \delta_{ij} &= 0, & i &\neq j, \\ &= 1, & i &= j, \end{aligned} \quad (2.2)$$

The exponential functions may be orthonormalized by a suitable linear combination, resulting in a set of so-called orthonormal exponentials. Geometrically, the orthonormal exponentials form a set of orthogonalized coordinates in a signal space which is indeed the same as that formed by ordinary exponentials.

As first pointed out by Kautz⁵, the orthonormalization of exponential function may be carried out in frequency domain much easier than in time domain. Given a set of exponential functions with poles at $-a_k \pm j\beta_k$, a set of orthonormal exponential functions may be constructed in the frequency domain as follows

$$\begin{aligned} \phi_{2k-1} &= B_{2k-1} \frac{s - \gamma_{2k-1}}{s^2 + p_k s + q_k} \prod_{i=1}^{k-1} \frac{s^2 - p_i s + q_i}{s^2 + p_i s + q_i} \\ \phi_{2k} &= B_{2k} \frac{s - \gamma_{2k}}{s^2 + p_k s + q_k} \prod_{i=1}^{k-1} \frac{s^2 - p_i s + q_i}{s^2 + p_i s + q_i}, \end{aligned} \quad (2.3)$$

where

$$\begin{aligned} p_k &= 2\alpha_k \\ q_k &= \alpha_k^2 + \beta_k^2 \end{aligned} \quad (2.4)$$

and B's are the normalization factors. The zeros, γ 's, can be chosen rather arbitrarily under the restriction of orthogonality in frequency domain that⁶

$$\int_{-j\infty}^{j\infty} \Phi_{2k-1}^*(-s) \Phi_{2k}(s) \frac{ds}{2\pi j} = 0. \quad (2.5)$$

In our case, this is equivalent to

$$\int_{-j\infty}^{j\infty} B_{2k-1} B_{2k} \frac{(-s - \gamma_{2k-1})}{(s^2 - p_k s + q_k)} \frac{(s - \gamma_{2k})}{(s^2 + p_k s + q_k)} \frac{ds}{2\pi j} = 0, \quad (2.6)$$

or

$$B_{2k-1} B_{2k} \frac{q_k + \gamma_{2k-1} \gamma_{2k}}{2p_k q_k} = 0. \quad (2.7)$$

Thus the only restriction on the zeros is

$$\gamma_{2k-1} \gamma_{2k} = -q_k. \quad (2.8)$$

Kautz⁵ chose the zeros in such a way that $\gamma_{2k-1} = \sqrt{q_k}$ and $\gamma_{2k} = -\sqrt{q_k}$. We decided to choose

$$\begin{aligned} \gamma_{2k-1} &= 0 \\ \gamma_{2k} &= \infty \end{aligned} \quad (2.9)$$

for simplicity of instrumentation. In this way, one function is the derivative of the other, and the resultant orthonormal exponential functions are

$$\begin{aligned}\Phi_{2k-1}(s) &= \frac{\sqrt{2p_k}s}{s^2+p_k s+q_k} \prod_{i=1}^{k-1} \frac{s^2-p_i s+q_i}{s^2+p_i s+q_i} \\ \Phi_{2k}(s) &= \frac{\sqrt{2p_k q_k}}{s^2+p_k s+q_k} \prod_{i=1}^{k-1} \frac{s^2-p_i s+q_i}{s^2+p_i s+q_i}.\end{aligned}\quad (2.10)$$

That Φ_{2k-1} and Φ_{2k} are orthogonal to all other orthonormal exponentials in the form of Equation (2.10) can be demonstrated by pole-zero cancellation⁷.

One advantage of orthonormal exponentials is the relative simplicity in constructing an orthonormal filter by cascading a number of filter sections as shown in Figure 1. If a signal $f(t)$ is applied to the input of the filter, the response at the k 'th terminal will be

$$c_k(t) = \int_0^\infty f(t-\tau) \varphi_k(\tau) d\tau \quad (2.11)$$

and the least-square approximation for the signal prior to the time t will be

$$f(t-\tau) \cong \sum_k c_k(t) \varphi_k(\tau), \quad (0 < \tau < \infty). \quad (2.12)$$

The error signal for the approximation prior to t will be

$$e(t, \tau) = f(t-\tau) - \sum_k c_k(t) \varphi_k(\tau) \quad (2.13)$$

which is a different function for different t . Obviously, if instead of $f(t)$, we apply to the input an impulse which is mathematically equivalent to Dirac's delta function, $\delta(t)$, the convolution integral shown in Equation (2.11) will then become

$$\int_0^{\infty} \delta(t - \tau) \varphi_k(\tau) d\tau = \varphi_k(t) \quad (2.14)$$

and the orthonormal filter serves as an orthonormal exponential generator.

Usually a physical signal $v(t)$ starts from $t = 0$ and extends to the future, and in order to approximate it in the least-square sense with ortho-normal exponentials, one has to evaluate the correlation integral,

$$c_k = \int_0^{\infty} v(\tau) \varphi_k(\tau) d\tau \quad (2.15)$$

We notice that there is a great similarity between the correlation integral and the convolution integral of the filter specified in Equation (2.11). If we let

$$f(t) = v(-t) \quad (2.16)$$

and evaluate the convolution integral at the instant $t = 0$, then Equation (2.12) become:

$$\begin{aligned} c_k(t) \Big|_{t=0} &= \int_0^{\infty} v(\tau-t) \varphi_k(\tau) d\tau \Big|_{t=0} \\ &= \int_0^{\infty} v(\tau) \varphi_k(\tau) d\tau \end{aligned} \quad (2.17)$$

which is precisely the correlation integral shown in Equation (2.15). As a result, the orthonormal filter serves the purpose of experimentally determining the coordinates of a signal $v(t)$ using orthonormal exponentials as basis functions. By applying the time-reversed signal $f(t)$ to the orthonormal filter, one can measure the coordinates from the output terminals at the instant corresponding to the epoch of the signal. This is the reason why we make the assumption that we will be concerned mainly with the time-reversed signal $f(t)$.

The realization of such an orthonormal filter with analog equipment is rather simple and will not be discussed, here. For digital computer realization, the well-known z -transform technique is required which shall be discussed in detail in the Appendix.

III. THEOREMS ON 'COMPLEMENTARY' OPERATOR

Since the 'complementary' operator (filter) plays a key role in our multiple-epoch problem, we shall discuss briefly some of the important theorems in this section. The 'complementary' operator for k pairs of exponential functions is defined as

$$G_k(s) = \prod_{i=1}^k \frac{s^2 - p_i s + q_i}{s^2 + p_i s + q_i} \quad (3.1)$$

Comparing this equation with Equation (2.10), we notice that the 'complementary' operator is inherent in the construction of orthonormal exponentials. To demonstrate its significance in orthogonalization of exponential functions, we assume a physically realizable signal

$$V(s) = \frac{s^2 - p_1 s + q_1}{s^2 + p_1 s + q_1} V'(s) \quad (3.2)$$

where $V'(s)$ may be any function (not necessarily exponentials) with the only limitation that its poles have to be in the left half plane. We shall show that $V(s)$ is orthogonal to $\Phi_1(s)$ and $\Phi_2(s)$. From Equation (2.10) and Equation (3.2), we have

$$\begin{aligned} \int_{-j\infty}^{j\infty} \Phi_1^*(-s) V(s) \frac{ds}{2\pi j} &= \int_{-j\infty}^{j\infty} \frac{-\sqrt{2p_1} s}{s^2 - p_1 s + q_1} \frac{s^2 - p_1 s + q_1}{s^2 + p_1 s + q_1} V'(s) \frac{ds}{2\pi j} \\ &= \int_{-j\infty}^{j\infty} \frac{-\sqrt{2p_1} s}{s^2 + p_1 s + q_1} V'(s) \frac{ds}{2\pi j} \quad (3.3) \end{aligned}$$

Notice that all the poles in the integrand are in the left half plane. It is analytic in the whole right half plane. Consequently

$$\int_{-j\infty}^{j\infty} \Phi_1(-s) V(s) \frac{ds}{2\pi j} = 0 \quad (3.4)$$

and $\Phi_1(s)$ and $V(s)$ are orthogonal. Similarly we can show that $\Phi_2(s)$ and $V(s)$ are also orthogonal. We have indeed proved the statement made in last section that Equation (2.10) represents a set of orthonormal functions.

Another important property is that the 'complementary' operator is a unitary operator. This can be easily shown by the definition shown in Equation (3.1) and by some simple algebraic manipulation that

$$G_k^*(-s) G_k(s) = 1 \quad (3.5)$$

which is indeed the definition of a unitary operator.

In addition to the properties stated above, there are some important theorems proven by Young and Huggins³. We shall simply state them as follows:

Theorem 1. With signal $f(t)$ as input, the output of the 'complementary' operator, $a(t)$, is simply the 'present-instant' error of the least-square approximation of $f(t)$ prior to the instant t . Mathematically, it may be expressed as

$$a(t) = e(t, \tau) \Big|_{\tau=0} = e(t, 0) \quad (3.6)$$

where $e(t, \tau)$ is the error signal defined in Equation (2.13). This theorem gives us a new insight into the properties of the 'complementary' operator.

Theorem 2. The integral of $|a(t)|^2$ up to the instant t is equal to the total error energy for the least-square approximation of $f(t)$ prior to the instant t . In other words

$$\int_{-\infty}^t |a(t)|^2 dt = E(t) \quad , \quad (3.7)$$

where

$$\begin{aligned} E(t) &\equiv \int_0^{\infty} |e(t, \tau)|^2 d\tau \\ &= \int_0^{\infty} |f(t-\tau)|^2 d\tau - \sum_k |c_k(t)|^2 \\ &= \int_{-\infty}^t |f(t)|^2 dt - \sum_k |c_k(t)|^2 \end{aligned} \quad (3.8)$$

Equation (3.8) is rather difficult to implement accurately, especially with analog equipment, because of the addition and subtraction of the squares of variables. It is obvious from Equation (3.7) that Theorem 2 provides us a much simpler way to evaluate the error energy. This is important for our present purpose since we use $\dot{E}(t)/E(t)$ as one of our criteria for epoch detection. Notice that if $f(t)$ can be exactly represented by the set of growing exponentials until the appearance of first epoch t_1 , then the output of the 'complementary' operator, $a(t)$, will be zero in the time interval $(-\infty, t_1)$. For otherwise, $E(t)$ would not be zero according to Equation (3.7), which is contradictory to our assumption that $f(t)$ prior to t_1 is exactly representable by the set of exponentials.

Theorem 3. If we assume the input signal $f(t)$ to be zero for $t > 0$, i. e.

$$\begin{aligned} f(t) &= f(t), & t < 0 & , \\ &= 0 & , & t > 0 & , \end{aligned} \quad (3.9)$$

then the output signal $a(t)$ may be separated into two parts such that

$$a(t) = a'(t) + a''(t) \quad (3.10)$$

and

$$\begin{aligned} a'(t) &= a(t), \quad t < 0, \\ &= 0, \quad t > 0 \\ a''(t) &= 0, \quad t < 0 \\ &= a(t), \quad t > 0 \end{aligned} \quad (3.11)$$

where

$$\begin{aligned} a'(t) &= \int_{-\infty}^{\infty} \epsilon(t-\tau, 0) g_k(\tau) d\tau, \\ a''(t) &= \int_0^{\infty} f_a(t-\tau) g_k(\tau) d\tau. \end{aligned} \quad (3.12)$$

$g_k(\tau)$ is the impulse response of the 'complementary' operator G_k , $f_a(t)$ is the least-square approximation of $f(t)$ and $\epsilon(t, 0)$ is the error signal of this approximation. In other words, $a(t)$ may be separated in two parts by a simple gate operation in time domain. The portion, $a'(t)$ of the signal prior to $t = 0$ is completely determined by the error signal, while the portion $a''(t)$ after $t = 0$ is completely determined by the approximating signal. Provided there exists an inverse operator, we shall be able to recover $\epsilon(t, 0)$ and $f_a(t)$ from them rather easily. For this reason the signal $a'(t)$ is called the 'complementary' error signal, $a''(t)$, the 'complementary' approximating signal and $a(t)$, the 'complementary' signal. This theorem is of particular importance to our case, as will be self-evident in the next section.

For a signal $f(t)$ which is not exactly representable by M exponentials, there exist a set of M exponential functions which is optimum in the least-square sense. McDonough⁴ has proven the following important theorem.

Theorem 4. For a certain set of M exponents, the approx-
imating signal energy will be maximum or minimum with respect
to all other choices of M exponents if and only if the 'complemen-
tary' error signal is orthogonal to the set of M exponential functions.
This theorem enables us to find the set of M exponents which is
optimum in the least-square sense by means of an iterative process
as we mentioned earlier.

IV. EPOCH DETECTION OF SIGNALS WITHOUT NOISE

The multiple-epoch signals discussed in this section are signals without noise. Each individual signal is assumed to be exactly representable by a set of exponential functions, or, at least, representable with relatively small error. Theoretically the error of the approximation of any signal may be reduced to as small as we wish by using sufficient number of exponential components. Thus the above assumption imposes no theoretical limitation, but in practice it does limit the class of signals to which our method is applicable.

The exponents of the set of exponential components that constitutes the individual signals may be known or unknown. If they are unknown, a modified Prony's method or McDonough's method may be applied to find out these exponents. Since the tail of the multiple-epoch signal consists of all the overlapping individual signals, the application of the modified Prony's method would give us the value of the exponents of all the exponential functions that constitute the individual signals. The problem is then reduced to the problem of signals comprising known exponential functions. We would like to remind the reader that the coefficients, c 's, of the individual signals on orthonormal exponentials are unknown. In other words, the signal space (or subspace) that contains all the individual signal vector is known, but the individual signal vector per se is unknown.

Let us first consider the simplest case of one signal which consists of m pairs of exponentials. The problem is to find the epoch of this single-epoch signal, and then to find the $2m$ coefficients of this signal on the $2m$ -dimensional orthonormal exponential basis. The m pairs of exponential functions are assumed known. Then one may evaluate as a function of the variable

t the error energy $E(t)$ which is the error energy of the least-square approximation of the signal $f(t)$ prior to the instant t . Thus the variable t may be considered as a tentative epoch under testing. Since the signal $f(t)$ consists of $2m$ exponential functions, it is well known from the property of exponentials that the signal in the interval $(-\infty, t)$ is also exactly representable by the same set of $2m$ exponentials. As a consequence,

$$\begin{aligned} E(t) &= 0, \quad t \leq t_1 \\ &> 0, \quad t > t_1 \end{aligned} \quad (4.1)$$

because there would be errors if one tries to approximate the signal $f(t)$ with a tentative epoch t such that $t > t_1$ where t_1 is the true epoch of the signal². The error energy will begin to increase at the true epoch t_1 . Indeed it should be obvious from Theorem 2 of last section, $E(t)$ is ever increasing (or non-decreasing); and as a result

$$\dot{E}(t) = \frac{dE(t)}{dt} \geq 0 \quad (4.2)$$

Now at the epoch t_1 , if $\dot{E}(t_1) > 0$, then according to Equation (4.1)

$$\frac{\dot{E}(t_1)}{E(t_1)} = \frac{\dot{E}(t_1)}{0} = \infty, \text{ for } E(t_1) = 0 \quad (4.3)$$

On the other hand, if $\dot{E}(t_1) = 0$, then the ratio $\dot{E}(t_1)/E(t_1)$ becomes indeterminate. However in that case since $E(t_1)$ is non-decreasing, $\dot{E}(t_1) \geq 0$; and if $\dot{E}(t_1) = 0$, $\ddot{E}(t_1) \geq 0$, etc. Consequently from Equation (4.3) and the well-known L'Hospital's rule for indeterminate form, the epoch t_1 has the unique property that

$$\frac{\dot{E}(t_1)}{E(t_1)} = \infty \quad (4.4)$$

It should be noted that this is not true for either $t < 0$ or $t > 0$. The reason for $t > 0$ is obvious since $E(t) > 0$ according to Equation (4.1). As for $t < 0$, although $E(t) = 0$, the time derivatives $\dot{E}(t)$, $\ddot{E}(t)$, are also zero, thus giving the unique property of the epoch t_1 as shown in Equation (4.4).

The reason of our choice of epoch detection criterion should be obvious from the above discussion. For signals which cannot be exactly represented by the set of exponentials, but have some relatively small error in the approximation, the epoch detection criterion will not give us an infinite value, but a maximum near the true epoch.

The evaluation of error energy may be simplified by taking advantage of the property of 'complementary' operator. The epoch detection criterion becomes

$$\frac{\dot{E}(t)}{E(t)} = \frac{a^2(t)}{\int_{-\infty}^t a^2(t) dt} \quad (4.5)$$

where $a(t)$ is the output at the 'complementary' operator with the signal $f(t)$ as input.

Next let us assume two overlapping signals $f_1(t)$ and $f_2(t)$ such that

$$\begin{aligned} f_1(t) &= f_1(t), \quad t \leq t_1, \\ &= 0, \quad t > t_1, \\ f_2(t) &= f_2(t), \quad t \leq t_2, \\ &= 0, \quad t > t_2, \end{aligned} \quad (4.6)$$

and

$$t_2 - t_1 \geq T_0 \quad (4.7)$$

The exponents of the exponentials constituting $f_1(t)$ and $f_2(t)$ are either known or evaluated. Suppose there are m_1 pairs of exponentials for $f_1(t)$ and m_2 pairs for $f_2(t)$ of which m_{12} pairs are common to both of them, then the 'complementary' operator is

$$G = \prod_{i=1}^{m_1} \frac{s^2 - p_i s + q_i}{s^2 + p_i s + q_i} \prod_{j=1}^{m_2 - m_{12}} \frac{s^2 - p_j s + q_j}{s^2 + p_j s + q_j} . \quad (4.8)$$

The 'complementary' signals are

$$\begin{aligned} a_1(t) &= 0, & t < t_1, \\ &= a_1(t), & t \geq t_1, \end{aligned} \quad (4.9)$$

and

$$\begin{aligned} a_2(t) &= 0, & t < t_2, \\ &= a_2(t), & t \geq t_2, \end{aligned}$$

in accordance with Equation (4.6) and Theorem 3. Therefore the total 'complementary' signal for our two-epoch signal

$$f(t) = f_1(t) + f_2(t) \quad (4.10)$$

is

$$\begin{aligned} a(t) &= 0, & -\infty < t < t_1, \\ &= a_1(t), & t_1 \leq t < t_2, \\ &= a_1(t) + a_2(t), & t_2 \leq t. \end{aligned} \quad (4.11)$$

Following an argument similar to the case of single-epoch case, we have

$$\frac{\dot{E}(t_1)}{E(t_1)} = \frac{a^2(t_1)}{\int_{-\infty}^{t_1} a^2(t) dt} = \infty. \quad (4.12)$$

It should be noted that the second epoch t_2 does not have this property because in the time interval (t_1, t_2) , the 'complementary' signal $a(t)$ is not zero as indicated in Equation (4.11). The generalization to n-epoch signal is obvious and we may state it in the following form.

Theorem 5. For a multiple-epoch signal for which each overlapping signal is exactly representable by a set of m pairs of exponentials, the first epoch t_1 has the unique property that $\dot{E}(t_1)/E(t_1)$ is infinite.

Let us return to the two-epoch problem and try to find a solution for detecting the second epoch. Referring to Equation (4.11), we notice that

$$a(t) = a_1(t), \quad t_1 < t < t_1 + T_0 < t_2, \quad (4.12)$$

under the assumption of (4.7). In other words, in the time interval $(t_1, t_1 + T_0)$, the 'complementary' signal is completely determined by the signal $f_1(t)$. Let us write abstractly

$$a(t_1 \leq t \leq t_1 + T_0) = O \left[f_1(-\infty < t \leq t_1) \right], \quad (4.13)$$

where O is an operator operated on the signal $f_1(t)$. The first epoch t_1 has been found by the epoch detection criterion, and as a result, $a(t_1 \leq t \leq t_1 + T_0)$ is known. If we are able to find an inverse operator O^{-1} , we may then recover the signal $f_1(t)$ by the relationship

$$f_1(-\infty < t \leq t_1) = O^{-1} \left[a(t_1 \leq t \leq t_1 + T_0) \right]. \quad (4.14)$$

We shall show that this is indeed possible, although O^{-1} is a rather complicated operation. Subtracting the recovered $f_1(t)$ from $f(t) = f_1(t) + f_2(t)$, the remaining is then reduced to the single-epoch detection problem. Again the generalization from two overlapping

signals to 3, 4, n is obvious. Thus, we have the following theorem.

Theorem 6. For a multiple-epoch signal for which each overlapping signal is exactly representable by a set of m pairs of exponentials, the 'complementary' signal $a(t)$ in the time interval $(t_1, t_1 + T_0)$ is completely determined by the signal $f_1(t)$ and an inverse operator may be found to recover $f_1(t)$ from $a(t)$ in the time interval $(t_1, t_1 + T_0)$. It is assumed that the nearest epoch t_2 is such that $t_2 - t_1 \geq T_0$.

The remaining part of this section is devoted to finding the inverse operator O^{-1} . It is noted that the operator O is nothing other than a 'complementary' operator G followed by a truncating operation operated at $t = T_0$. It is precisely this truncating operation which causes some complications in deriving the inverse operator O^{-1} , since otherwise a simple inversion of the 'complementary' operator G would give us the desired result.

The signal $f_1(t)$ consists of growing exponentials, thus it may be written as

$$f_1(t) = \sum_{k=1}^m \left\{ c_{2k-1} \phi_{2k-1}(-t) + c_{2k} \phi_{2k}(-t) \right\}, \quad (4.15)$$

since ϕ_{2k-1} and ϕ_{2k} are defined as damped orthonormal exponentials in the time interval $(0, \infty)$. In frequency domain,

$$F_1(s) = \sum_{k=1}^m \left\{ c_{2k-1} \Phi_{2k-1}^*(-s) + c_{2k} \Phi_{2k}^*(-s) \right\}, \quad (4.16)$$

where the notation "*" indicates complex conjugate, and $\Phi(s)$ are defined in Equation (2.10). Since $a_1(t)$ is the output of the 'complementary' filter with $f_1(t)$ as input, its frequency domain representative may be written as

$$A_1(s) = \sum_{k=1}^m \left\{ c_{2k-1} U_{2k-1}(s) + c_{2k} U_{2k}(s) \right\}, \quad (4.17)$$

where

$$U_{2k-1}(s) = G_m(s) \Phi_{2k-1}^*(-s)$$

$$= \frac{-\sqrt{2p_k} s}{s^2 + p_k s + q_k} \prod_{i=k+1}^m \frac{s^2 - p_i s + q_i}{s^2 + p_i s + q_i}$$

$$U_{2k}(s) = G_m(s) \Phi_{2k}^*(-s)$$

$$= \frac{\sqrt{2p_k q_k}}{s^2 + p_k s + q_k} \prod_{i=k+1}^m \frac{s^2 - p_i s + q_i}{s^2 + p_i s + q_i}. \quad (4.18)$$

It is interesting to note that the set of function $U(s)$ are orthonormal exponentials, and furthermore, they are damped exponentials as it should be.

For the two-epoch problem, we can only obtain at the output of 'complementary' operator $a(t)$ rather than $a_1(t)$. The available information of $a_1(t)$ is contained in the time interval $(t_1, t_1 + T_0)$ according to Equation (4.12). Thus

$$a(t) = \sum_{i=1}^{2m} c_i u_i(t - t_1)$$

$$\text{for } t_1 \leq t \leq t_1 + T_0 \quad (4.19)$$

The time translation of $u_i(t)$ (which corresponds to $U_i(s)$) is necessary for the coincidence of origin. We wish to find c_i 's from the known basis function u_i 's and $a(t)$ in the time interval

where ξ_k, ξ_k^* are complex conjugates as functions of t which may be measured, then the deterministic signal at the instant $t + \tau$ will be

$$f_g(t + \tau) = h_g(t + \tau; t), \quad t \leq t_1, \\ t + \tau \leq t_1. \quad (5.3)$$

$h_g(t + \tau; t)$ is defined as the extrapolated signal at the instant $t + \tau$ based on the measurement at t . τ may be positive or negative depending on whether we are interested in extrapolating the future value or the past value. Mathematically it may be written as

$$h_g(t + \tau, t) = \sum_{k=1}^m \left\{ \xi_k(t) e^{(a_k + j\beta_k)(t + \tau)} + \xi_k^*(t) e^{(a_k - j\beta_k)(t + \tau)} \right\}. \quad (5.4)$$

Here we deal with ordinary exponentials rather than orthonormal exponentials for the clarity of presentation. The two forms are related to each other by a simple linear transformation.

That Equation (5.3) holds true is the result of continuity of the deterministic signal $f_g(t)$ and all its derivatives for $-\infty < t < t_1$. The epochs have the unique property that it is the continuation of the past while it marks the beginning of discontinuity as shown in Figure 2A. This property may indeed be considered as the definition of an epoch. This property is not clearly visible when the discontinuity is in the high order derivatives. It is also masked when the signal is corrupted by random noise. However, a likelihood ratio criterion based on this property gives us remarkable results for epoch detection.

The likelihood test is well-known in statistical communication theory for the test of hypotheses. One of the simplest hypothesis

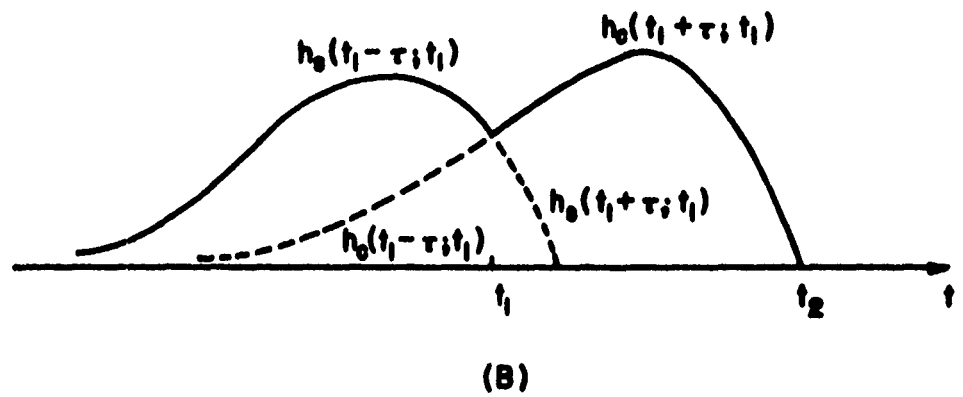
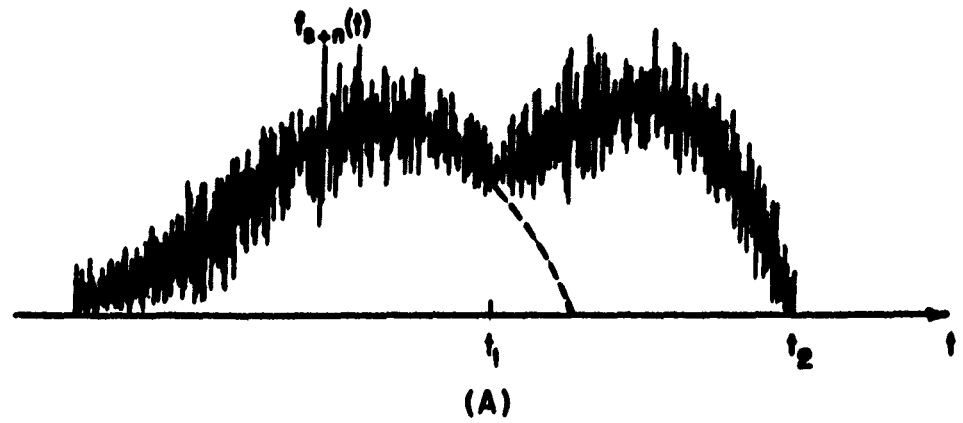


FIGURE 2 EXTRAPOLATION AT THE EPOCH t_1 .

tests is to test the hypothesis H_0 that certain observations are distributed according to a Gaussian distribution with mean m_0 and variance σ^2 , against the alternative hypothesis H_1 which is a Gaussian distribution with mean m_1 and variance σ^2 . Based on the level of the test, a threshold η may be set for the likelihood ratio

$$\frac{P_1(y)}{P_0(y)} = \exp \left[-\frac{1}{2\sigma^2} (y-m_1)^2 + \frac{1}{2\sigma^2} (y-m_0)^2 \right]. \quad (5.5)$$

The level of test is in turn pre-determined in accordance with the cost of rejecting H_0 when in fact H_0 is true and that of accepting H_0 when in fact H_1 is true. If

$$\frac{P_1(y)}{P_0(y)} \geq \eta, \quad (5.6)$$

we then reject the hypothesis H_0 , and assume that H_1 is true.

To adapt this idea to our problem, we notice that our signal is corrupted with random noise so that it may be written as

$$f_{s+n}(t) = f_s(t) + f_n(t) + f_n(t), \quad (5.7)$$

where $f_n(t)$ is the random Gaussian noise. The random noise is of zero mean and unknown variance. Consequently, the noisy signal $f_{s+n}(t)$ may be regarded as a random distribution with the deterministic signal $f_s(t)$ as its mean. In other words, for the noisy signal $f_{s+n}(t)$, the mean value varies with time according to the variation of the deterministic signal $f_s(t)$, while the randomness is completely contributed by the noise $f_n(t)$. For the unknown variance, we may simply assign to it a value σ_n^2 . The noise is assumed independent of the deterministic signal, so that the measurements of orthonormal filters at any

instant t are not affected (actually, there is a very small effect) by the noise, nor are the extrapolations. Indeed, we shall not make the distinction between the extrapolation based on the noisy signal $f_{s+n}(t)$ and that of deterministic signal $f_s(t)$. The notation $h_s(t + \tau; t)$ may be interpreted as extrapolated signal at the instant $t + \tau$ based on the measurements of the noisy signal $f_{s+n}(t)$ at the instant t .

The noisy signal $f_{s+n}(t)$ may be regarded as a particular member of an ensemble of multi-dimensional gaussian distribution. Suppose we sample the signal N instants, $f_{s+n}(t_i)$ where $i = 1, 2, \dots, N$, the probability of this particular member of an ensemble with mean $f_s(t_i)$ is then

$$P = \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^N \left[f_{s+n}(t_i) - f_s(t_i) \right]^2 \right\} . \quad (5.8)$$

It is more convenient to express the probability in terms of the logarithm. Thus, as N approaches infinity, we have in the limit

$$\log P = -\frac{1}{2\sigma^2} \int \left[f_{s+n}(t) - f_s(t) \right]^2 dt . \quad (5.9)$$

It is indeed unnecessary to take the limit since we shall use a digital computer for calculation, and the signal has to be sampled. Nevertheless the integral expression does facilitate our following discussion.

As we mentioned earlier, the epoch has the unique property that it is the continuation of the past while it marks the beginning of discontinuity. To make it more clear, let us assume a two-epoch deterministic signal as shown in Figure 2 such that

$$\begin{aligned}
 f_{s1}(t) &= f_{s1}(t), & t \leq t_1, \\
 &= 0, & t > t_1, \\
 f_{s2}(t) &= f_{s2}(t), & t \leq t_2, \\
 &= 0, & t < t_2;
 \end{aligned} \tag{5.10}$$

$$t_2 - t_1 \geq T_0 \tag{5.11}$$

and

$$\begin{aligned}
 f_{s+n}(t) &= f_s(t) + f_n(t) \\
 &= f_{s1}(t) + f_{s2}(t) + f_n(t)
 \end{aligned} \tag{5.12}$$

Consider the first epoch t_1 , since it is the continuation of the past

$$f_s(t_1) = h_s(t_1 - \tau; t), \quad \tau > 0, \tag{5.13}$$

as we discussed at the beginning of this section. Because of the discontinuity of the deterministic signal or of its derivatives, the relationship is no longer true for extrapolation into the future, viz.,

$$f_s(t_1 + \tau) \neq h_s(t_1 + \tau; t_1), \quad \tau > 0 \tag{5.14}$$

One would wonder whether there is a way of extrapolation which is equal to $f_s(t_1 + \tau)$. Indeed there is. Recalling Theorem 6, we have an operator O^{-1} which enable us to recover $f_{s1}(t)$ from the 'complementary' signal $a(t)$ in the time interval $t_1 \leq t \leq t_1 + T_0$. Subtracting $f_{s1}(t)$ from $f_s(t)$ gives us $f_{s2}(t)$. As a result, there is an extrapolation $h_c(t_1 \pm \tau; t_1)$ which is based on the measurements at t_1 and the 'complementary' signal $a(t)$ for $t_1 \leq t \leq t_1 + T_0$ such that

$$h_c(t_1 \pm \tau, t_1) = f_{s2}(t_1 \pm \tau),$$

$$t_1 + \tau \leq t_2 \tag{5.15}$$

Since, according to Equation (5.10), (5.11) and (5.12),

$$\begin{aligned} f_s(t) &= f_{s1}(t) + f_{s2}(t), \quad t \leq t_1 \\ &= f_{s2}(t), \quad t_1 < t \leq t_1 + T_0 < t_2 \end{aligned} \quad (5.16)$$

we have the important relationships that

$$f_s(t_1 - \tau) \neq h_c(t_1 - \tau; t_1), \quad 0 < \tau \quad (5.17)$$

and

$$f_s(t_1 + \tau) = h_c(t_1 + \tau; t_1), \quad 0 < \tau \leq T_0 \quad (5.18)$$

The four Equations, (5.13), (5.14), (5.17), and (5.18) mathematically describe the property of the first epoch t_1 of a two-epoch signal. (See Figure 2). They are also true for a n-epoch signal. Thus we have:

Theorem 7. For a multiple-epoch deterministic signal consisting of overlapping signals which are individually exactly representable by a set of exponential functions, the first epoch t_1 has the properties described by Equations (5.13), (5.14), (5.17) and (5.18).

It should be noted that since we assume the noise to be independent of the deterministic signal, the extrapolations, $h_s(t_1 \pm \tau; t_1)$ and $h_c(t_1 \pm \tau; t_1)$ are obtainable from the measurements of the noisy signal $f_{s+n}(t)$ at t_1 . We may similarly obtain $h_s(t \pm \tau; t)$ and $h_c(t \pm \tau; t)$ for any instant t where $h_c(t \pm \tau; t)$ is defined by the complex operation consisting of the 'complementary' operator, the inverse operator O^{-1} , the measurements of the orthonormal filter at the instant t , and extrapolation based on them. The extrapolations has the properties such that

$$f_s(t \pm \tau) = h_s(t \pm \tau; t) = h_c(t \pm \tau; t) \quad (5.19)$$

for $\tau \leq T_0$ and $t \leq t_1 - T_0$,

where we have used T_0 as a basic time unit. The first identity is obvious because of the continuity of the deterministic signal and all its derivatives in that time interval. As for the second identity in Equation (5.19), it is because of the independence of the noise from the exponential functions, so that

$$O^{-1} \left[a(t \leq t' \leq t + T_0) \right] = 0 \quad (5.20)$$

for $t \leq t_1 - T_0$,

although $a(t')$ in that interval is not absolutely zero but consists of noise. Consequently, there is nothing to be subtracted from the original noisy signal and the extrapolation $h_c(t \pm \tau; t)$ is equivalent to $h_s(t \pm \tau; t)$. On the other hand for t much later than t_1 we have,

$$\begin{aligned} f_s(t \pm \tau) &\neq h_s(t \pm \tau; t) \\ &\neq h_c(t \pm \tau; t) \end{aligned} \quad (5.20)$$

for $\tau \leq T_0$ and $t \geq t_1 + T_0$.

A transition region of course exists between $t_1 - T_0$ and $t_1 + T_0$, with the epoch t_1 having the important property prescribed in Theorem 7.

Returning to hypothesis testing of mean, the mean value of our noisy signal in the time interval $(t, t + T_0)$ may be $h_s(t+\tau; t)$, $h_c(t+\tau; t)$ or none of them. At the epoch t_1 , the noisy signal should have a mean which equals to $h_c(t+\tau; t)$ according to (5.18). As a result, the logarithm of a likelihood defined as

$$\begin{aligned} \text{Log } \xi_f &= \frac{-1}{2\sigma^2} \int_0^{T_0} \left[f_{s+n}(t+\tau) - h_c(t+\tau; t) \right]^2 d\tau \\ &+ \frac{1}{2\sigma^2} \int_0^{T_0} \left[f_{s+n}(t+\tau) - h_s(t+\tau; t) \right]^2 d\tau \end{aligned} \quad (5.21)$$

should have a relatively large positive number at t_1 . Similarly one may define a likelihood ratio based on extrapolation over the past as

$$\log \zeta_p = \frac{-1}{2\sigma^2} \int_0^{T_0} [f_{s+n}(t-\tau) - h_s(t-\tau; t)]^2 d\tau + \frac{1}{2\sigma^2} \int_0^{T_0} [f_{s+n}(t-\tau) - h_c(t-\tau; t)]^2 d\tau, \quad (5.22)$$

which again should be a positive number for $t = t_1$ in accordance with Equation (5.13) and (5.17). One may also define

$$\log \zeta_s = \frac{-1}{2\sigma^2} \int_0^{T_0} [f_{s+n}(t-\tau) - h_s(t-\tau; t)]^2 d\tau + \frac{1}{2\sigma^2} \int_0^{T_0} [f_{s+n}(t+\tau) - h_s(t+\tau; t)]^2 d\tau \quad (5.23)$$

and

$$\log \zeta_c = -\frac{1}{2\sigma^2} \int_0^{T_0} [f_{s+n}(t+\tau) - h_c(t+\tau; t)]^2 d\tau + \frac{1}{2\sigma^2} \int_0^{T_0} [f_{s+n}(t-\tau) - h_c(t-\tau; t)]^2 d\tau. \quad (5.24)$$

Again these two functions should have relatively large values at $t = t_1$. For $t \leq t_1 - T_0$, theoretically all these four equations reduce to zero. As to the case when $t \geq t_1 + T_0$, the situation becomes obscure. However, we intuitively expect that in general, the values of above four equations for $t \leq t_1 + T_0$ should be smaller than at $t = t_1$, simply because for the former, the mean of f_{s+n} follows neither of the extrapolations, while for $t = t_1$, in all four likelihood ratio equations the mean of f_{s+n} should equal to one of the extrapolations but not the other. This does not exclude the possibility that at a certain instant $t' > t_1 + T_0$, a certain likelihood ratio is larger than at t_1 , since our problem is statistical in nature. Furthermore there is the transition interval $t_1 - T_0 \leq t \leq t_1 + T_0$.

To minimize the above-mentioned possibilities which would cause the detection of a wrong epoch, we set up a threshold

$$\eta = 0 \quad (5.24)$$

such that an epoch must satisfy the four necessary conditions, namely

$$\begin{aligned} \log \zeta_f &\geq \eta \\ \log \zeta_p &\geq \eta \\ \log \zeta_s &\geq \eta \\ \log \zeta_c &\geq \eta \end{aligned} \quad (5.25)$$

These inequalities are not sufficient, and we detect the epoch by the criterion

$$\log \zeta(t_{\max}) = \max \quad (5.26)$$

among all possible epochs, where

$$\zeta \equiv \zeta_f \times \zeta_p \equiv \zeta_s \times \zeta_c \quad (5.27)$$

is a function of t . Equation (5.25) and (5.26) summarize our method of detecting epoch.

To detect the second epoch t_2 after the detection of t_1 , we again utilize Theorem 6 to recover $f_1(t)$ and reduce the problem into the same form of detecting epoch t_1 . The generalization into the detection of t_3, t_4, \dots, t_n is obvious by iteration of this same process.

We would like to say a few words about the variance. We notice that in the likelihood ratio equations, σ^2 only serves as a scale factor. It will have no effect on Equation (5.25) since we assume $\eta = 0$. A change of the value of variance will change the values of the likelihood ratio, but it certainly will not change the value of t_{\max} which corresponds to maximum value of $\zeta(t)$. If we are interested

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in a threshold other than zero, however, Equation (5.25) is certainly affected by the value σ^2 , and a knowledge of the variance of the random noise $f_n(t)$ becomes important.

Sinusoidal signals are particular cases of exponential signals, and as such the procedure described in this section should be applicable to multiple-epoch sinusoidal signals. Certain modifications are required for the procedure mainly because the poles of sinusoidal signals are located on the imaginary axis rather than in the left-half plane. We shall not go into detail for the procedure for sinusoidal signals, but simply say that it is much simpler than the exponential signals.

VI. EXPERIMENTAL RESULTS

Let us summarize the procedure for the detection of multiple-epoch signals.

1) The exponents of exponential functions that form our deterministic signal are assumed known or have been estimated from Prony's method or McDonough's method.

2) An orthonormal filter is constructed to measure the noisy signal $f_{s+n}(t)$ for each instant t .

3) Likelihood ratios are calculated for each instant. Based on the criteria in Equation (5.35) and (5.26), the first epoch t_1 may be detected.

4) The deterministic signal $f_{s1}(t)$ which corresponds to epoch t_1 is then obtainable from the 'complementary' signal, and may be subtracted from the noisy signal $f_{s+n}(t)$.

5) The second epoch may be detected in the same way as the first epoch from the signal $f_{s+n}(t) - f_{s1}(t)$.

6) Similarly, t_3, t_4, \dots, t_n may be detected and the deterministic signal $f_s(t)$ is properly represented.

With this procedure, a multiple-epoch signal with noise shown in Figure 3 is represented by exponential function. The individual deterministic signals are exactly representable by exponential functions and the exponents are known. Figure 3B, 3C, and 3D are the logarithm of the likelihood ratios $\log \zeta$ for the detection of 1st, 2nd, and 3rd epoch respectively. Figure 3E, 3F, 3G are the individual signals measured and Figure 3H is the sum of the three. Figure 4 is similar to Figure 3 except that the exponential functions used to represent the overlapping signals are not exactly matched to the deterministic signals. They are $e^{1.9t}$ and $e^{1.6t}$ compared with e^{2t} and e^t which constitute the original signals, thus showing that our method does not require that the component functions match the signal exactly. The remarkable results are self-evident.

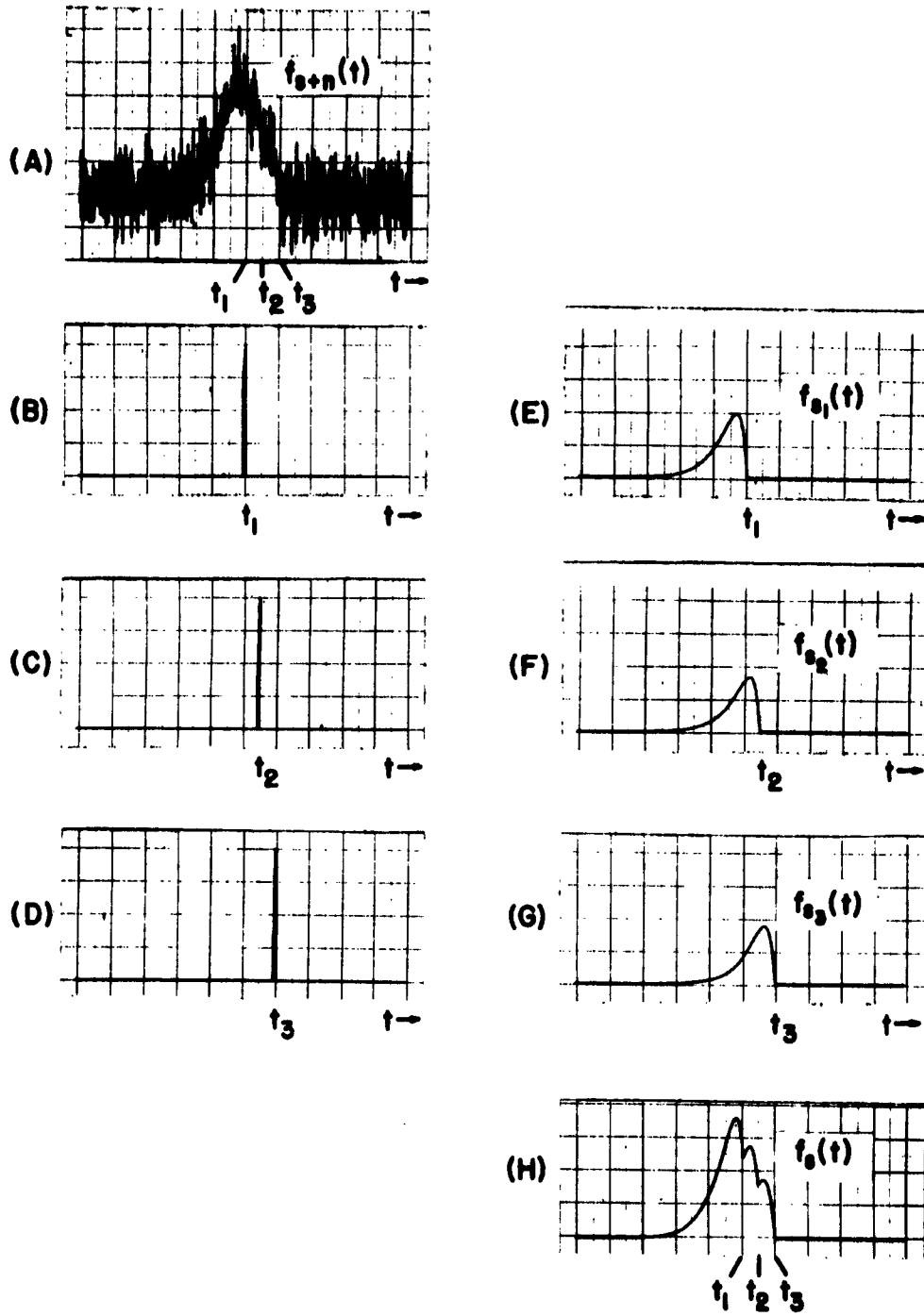


FIGURE 3 THE DETECTION OF EPOCHS.

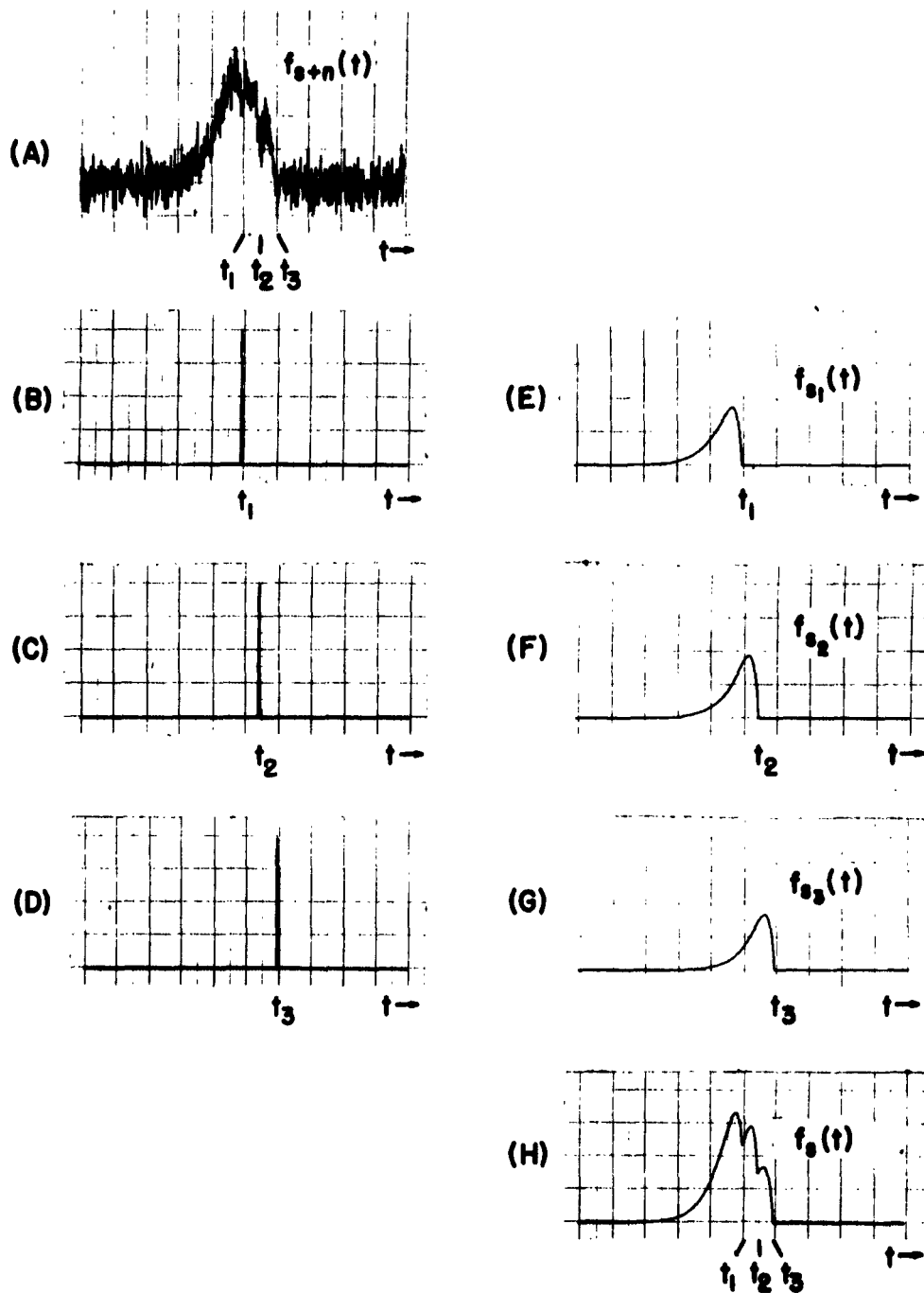


FIGURE 4 THE DETECTION OF EPOCHS.

We observed an interesting result from these two examples that the epochs could be detected by the likelihood ratio ζ_c alone which was defined in Equation (5.24). This is probably due to the particular waveshape of our signal. The reason is not clear to us at the moment and deserves further investigation.

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APPENDIX - Discrete Orthonormal Exponentials¹⁰

Since the procedure of multiple-epoch signal representation is most conveniently carried out on a digital computer, we shall deal with orthonormal exponentials in discrete form (that is, sampled data form). We notice that the sampled data form of continuous orthonormal exponentials which are mathematically expressed by their z-transforms, are not orthogonal to each other in z-domain. The z-transform of a signal involves the values of the signal at equal-spaced instants separated by the sampling interval

$$T = \frac{2\pi}{\omega_s}, \quad (A.1)$$

where ω_s is the sampling frequency. According to the sampling theorem, only a signal having a frequency spectrum lower than ω_s is exactly reproducible from the sampled data by the cardinal function. However the orthonormal exponentials which vanish identically over half of the t axis have a non-zero spectrum extending over almost all frequencies. As a result, orthogonality in the s-domain does not yield orthogonality in z-domain representatives of these continuous functions. To avoid this difficulty, we choose at the outset, discrete basis functions which are orthogonal in the z-domain corresponding to the s-domain poles of the continuous orthonormal exponentials. The zeros are different, however, although they are chosen in such a way that as the sampling interval approaches zero, our chosen basis functions will approach the s-domain orthonormal exponentials.

Given a function $f(t)$, the sampled values of this function at the sampling interval may be written as

$$f^s(t) = f(t) \times i(t), \quad (A.2)$$

where $i(t)$ represents a series of unit impulses equally spaced in time¹²

$$i(t) = \sum_{n=0}^{\infty} \delta(t-nT) \quad (A. 3)$$

The frequency-domain representative of this train of unit impulses is

$$\begin{aligned} I(s) &= 1 + e^{-sT} + e^{-2sT} + \dots \\ &= \frac{1}{1-e^{-sT}} \end{aligned} \quad (A. 4)$$

Suppose the sampled value of the signal $f(t)$ at the n 'th sampling instant is f_n , then

$$f^s(t) = \sum_n f_n \delta(t-nT) \quad (A. 5)$$

and

$$F^s(s) = \sum_n f_n e^{-nsT}$$

Define

$$z \equiv e^{sT} \quad (A. 6)$$

then

$$F^s(z) \equiv F^s(s) = \sum_n f_n z^{-n} \quad (A. 7)$$

For example, consider a simple exponential signal $\exp(-at)$ for $t \geq 0$; its sampled value will be

$$\begin{aligned} &1 + e^{-aT} z^{-1} + e^{-2aT} z^{-2} + \dots \\ &= \frac{1}{1-e^{-aT} z^{-1}} \end{aligned} \quad (A. 8)$$

One should notice that the z -transformation is a linear transformation,¹³ thus, if

$$f(t) = \sum_k c_k e^{s_k t} \quad , \quad (A.9)$$

which is equivalent to

$$F(s) = \sum_k \frac{c_k}{s - s_k} \quad ; \quad (A.10)$$

then, by Equation (A.8)

$$F^s(z) = \sum_k \frac{c_k}{1 - e^{s_k T} z^{-1}} \quad . \quad (A.11)$$

Since c_k is the residue of $F(s)$ with a pole at $s = s_k$, it follows from Cauchy's integral theorem that

$$F^s(z) = \oint_c \frac{F(\lambda)}{1 - e^{\lambda T} z^{-1}} \frac{d\lambda}{2\pi j} \quad , \quad (A.12)$$

where the contour encloses those singularities belonging to $F(s)$ only. A similar argument gives us the inverse transformation

$$F(s) = \int_{c'} \frac{F^s(z)}{s - (1/T) \ln z} \frac{dz}{2\pi j z} \quad (A.13)$$

where the contour c' encloses the poles of $F^s(z)$ only.

With these two transformations between $F(s)$ and $F^s(z)$ derived, the z -transforms of the continuous orthonormal exponentials can be easily found. Take the first pair of orthonormal exponential functions which, according to Equation (2.10), are

$$\begin{aligned} \Phi_1(s) &= \frac{\sqrt{ps}}{s^2 + ps + q} \\ \Phi_2(s) &= \frac{\sqrt{2pq}}{s^2 + ps + q} \end{aligned} \quad (A.14)$$

where we have used p, q instead of p_1, q_1 for simplicity. Substituting into Equation (A.12), we have

$$\Phi_1^s(z) = \sqrt{2p} \frac{z^2 - ze^{-\alpha T} \left[(\alpha/\beta) \sin \beta T + \cos \beta T \right]}{z^2 - 2ze^{-\alpha T} \cos \beta T + e^{-2\alpha T}} \quad (A.15)$$

$$\Phi_2^s(z) = \sqrt{2pq} \frac{ze^{-\alpha T} (1/\beta) \sin \beta T}{z^2 - 2ze^{-\alpha T} \cos \beta T + e^{-2\alpha T}}$$

where $\underline{\alpha}$ and $\underline{\beta}$ are defined as

$$\begin{aligned} \alpha &= \frac{p}{2} \\ \beta &= \sqrt{q - (p/2)^2} \end{aligned} \quad (A. 16)$$

which are the same as Equation (2.4).

These equations would appear to yield the sampled-date equivalent of the orthonormal exponentials. However a more detailed investigation of Equation (A.15) reveals that $\Phi_1^s(z)$ and $\Phi_2^s(z)$ are not orthogonal. Indeed they are not even normalized. One would have great difficulty in using these un-orthonormal functions for signal representation.

The reason that $\Phi_1^s(z)$ and $\Phi_2^s(z)$ are not orthonormal is as follows: Referring to Equation (A.12), we notice that the contour is taken in such a way that the integral represents the sum of the residues of the integrand at the poles of $F(\lambda)$. The result will be the same except for a sign change, if we take the residues of the other part of the integrand. In other words, for the integrand of Equation (A.12).

$$\frac{F(\lambda)}{1 - e^{\lambda T} z^{-1}} = \frac{F(\lambda)}{1 - e^{(\lambda - s)T}} \quad (A. 17)$$

The poles, excluding those of $F(\lambda)$, will be

$$\lambda = s + \frac{2\pi nj}{T} \quad (A. 18)$$

with

$$n = 0, \pm 1, \pm 2, \dots$$

Thus Equation (A.12) may be written as

$$\begin{aligned}
 F^s(z) &= - \sum_{n=-\infty}^{\infty} \left. \frac{F(\lambda)}{\frac{d}{d\lambda} 1 - e^{(\lambda-s)T}} \right|_{\lambda = s + \frac{2\pi nj}{T}} \\
 &= \frac{1}{T} \sum_{n=-\infty}^{\infty} F(s + jn\omega_s) \quad , \quad (A.19)
 \end{aligned}$$

where

$$\omega_s = \frac{2\pi}{T}$$

is the sampling frequency*. Assume that signal $f(t)$ has a frequency spectrum as shown in Figure 5(a), the sampled signal has a spectrum reproduced without modification provided $\omega_a < \omega_s/2$. On the other hand, if $\omega_a > \omega_s/2$, distortion arises due to the overlapping of the spectra separated ω_s apart¹⁴. In order to recover the signal from the sampled data, it is necessary that

$$\omega_s \geq 2\omega_a \quad , \quad (A.20)$$

or it may be written as

$$2WT \leq 1 \quad (A.21)$$

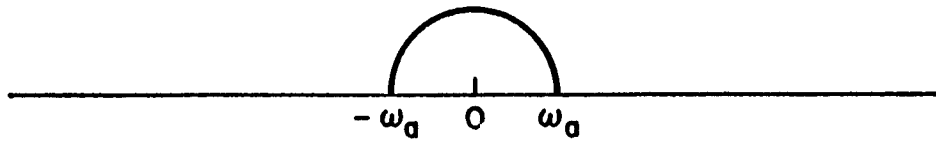
where

$$W = \frac{\omega_a}{2\pi} \quad . \quad (A.22)$$

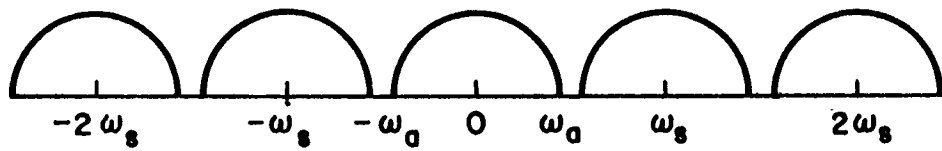
* Strictly speaking, Equation (A.19) should be

$$F^s(z) = F^s(s) = \frac{1}{T} \sum_{n=-\infty}^{\infty} F(s + jn\omega_s) + \frac{1}{2} f(0^+) \quad .$$

Spectrum of $f(t)$



Spectrum of $f^s(t)$ for $\omega_s > 2\omega_d$



Spectrum of $f^s(t)$ for $\omega_s < 2\omega_d$

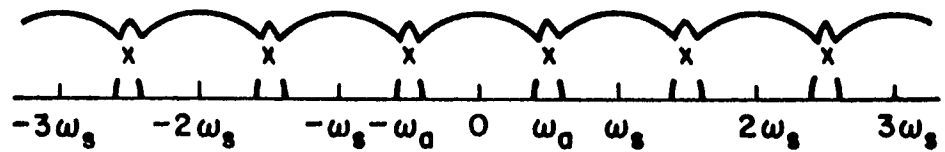


FIGURE 5 SPECTRUM OF SAMPLED-DATA SIGNAL.

This is indeed the sampling theorem¹¹. We would like to remind the reader that except for pure sinusoidal signals, orthonormal exponentials have non-zero spectra extending to infinity, and consequently, Equation (A.20) can never be satisfied. As a result, the z-transforms shown in Equation (A.15) are distorted representatives of orthonormal exponentials, and hence not orthogonal themselves.

From the Parseval relation, one realizes that in the frequency domain, the orthonormal condition for two functions ψ_i and ψ_j is

$$\int_{-j\infty}^{j\infty} \psi_i^*(-s) \psi_j(s) \frac{ds}{2\pi j} = \delta_{ij} \quad , \quad (\text{A.23})$$

where δ_{ij} is the Kronecker delta. By the definition of \underline{z} shown in Equation (A.6) so that

$$dz = T e^{sT} ds \quad , \quad (\text{A.24})$$

Equation (A.23) may be transformed into the form of

$$\oint \left[\psi_i^s(1/z) \right]^* \psi_j^s(z) \frac{dz}{2\pi j z T} = \delta_{ij} \quad , \quad (\text{A.25})$$

which is indeed the condition of orthonormality in z-domain. Notice that in Equation (A.25), the contour of integration is along the unit circle.

Let us take a pair of complex poles \underline{z}_1 and \underline{z}_1^* , and assume that there exist a pair of discrete orthonormal functions

$$\begin{aligned} \psi_1^s(z) &= b_1 \frac{z^2 - \mu_1 z}{(z - \underline{z}_1)(z - \underline{z}_1^*)} \\ \psi_2^s(z) &= b_2 \frac{z^2 - \mu_2 z}{(z - \underline{z}_1)(z - \underline{z}_1^*)} \quad , \end{aligned} \quad (\text{A.26})$$

where μ_1 and μ_2 are real numbers, and b_1 and b_2 are normalization constants. The condition of orthogonality requires that

$$\oint \left[\psi_1^s(1/z) \right]^* \psi_2^s(z) \frac{dz}{2\pi j z T}$$

$$= b_1^* b_2 \frac{(1+\mu_1\mu_2)(1+|z_1|^2) - (\mu_1+\mu_2)(z_1+z_1^*)}{T(1-|z_1|^2)(1-z_1^2)(1-z_1^{*2})}$$

$$= 0 \quad , \quad (A.27)$$

or that

$$(1+\mu_1\mu_2)(1+|z_1|^2) - (\mu_1+\mu_2)(z_1+z_1^*) = 0 \quad . \quad (A.28)$$

Therefore one may choose any value of μ_1 and μ_2 as long as they satisfy Equation (A.28). One simple choice is to let

$$\begin{aligned} \mu_1 &= 1 \\ \mu_2 &= -1 \end{aligned} \quad (A.29)$$

which does satisfy Equation (A.28). Then

$$\psi_1^s(z) = b_1 \frac{z^2 - z}{(z - z_1)(z - z_1^*)}$$

$$\psi_2^s(z) = b_2 \frac{z^2 + z}{(z - z_1)(z - z_1^*)} \quad , \quad (A.30)$$

where the normalization constant can be evaluated easily as

$$b_1 = \left\{ \frac{T}{2} (1 - |z_1|^2) \left[1 + (z_1 + z_1^*) + |z_1|^2 \right] \right\}^{1/2}$$

$$b_2 = \left\{ \frac{T}{2} (1 - |z_1|^2) \left[1 - (z_1 + z_1^*) + |z_1|^2 \right] \right\}^{1/2}. \quad (\text{A. 31})$$

For a pair of s -domain poles

$$\begin{aligned} s_1 &= -\alpha + j\beta \\ s_1^* &= -\alpha - j\beta \end{aligned} \quad (\text{A. 32})$$

the corresponding z -domain poles will be

$$\begin{aligned} z_1 &= e^{s_1 T} = e^{-\alpha T + j\beta T} \\ z_1^* &= e^{s_1^* T} = e^{-\alpha T - j\beta T} \end{aligned}$$

Then the poles of the ψ 's shown in Equation (A. 30) are indeed the same as the poles of the Φ 's in Equation (A. 15). Furthermore, as the sampling interval T approaches zero, the zeros of Φ_1 become the same as that of ψ_1 . If instead of $\sqrt{2p}$, we assign to $\Phi_1^s(z)$ a normalization constant in the z -domain, then

$$\lim_{T \rightarrow 0} \Phi_1^s(z) = \psi_1^s(z) \quad (\text{A. 34})$$

Since $\Phi_1^s(z)$ is the sampled value of a continuous exponential function $\varphi_1(t)$, $\psi_1^s(z)$ approaches the z -transform of $\varphi_1(t)$ as the sampling interval T approaches zero. The relation between $\psi_2^s(z)$ and $\Phi_2^s(z)$ is not so obvious due to the peculiarity of the zero of ψ_2 at $z = 1$ which corresponds to the zeros at $\pm j\omega_s/2$ in s -domain. However, owing to the location of the poles of Φ 's and ψ 's, Equation (A. 15) and Equation (A. 30) indeed describe the same two-dimensional signal space. Since $\varphi_1(t)$ and $\varphi_2(t)$ are orthogonal, and so are $\psi_1^s(z)$ and $\psi_2^s(z)$, our conclusions are that as T approaches zero, $\psi_2^s(z)$ approaches the z -transforms of the continuous orthonormal exponential $\varphi_2(t)$.

As we discussed in Section III, the 'complementary' operator plays a central role in the orthogonalization of continuous exponential functions. This is also true for the discrete case. In fact, all the theorems concerning the 'complementary' operator may easily be carried over to the discrete case. The 'complementary' operator in z -domain may be written as,

$$G_k^s(z) = \prod_{i=1}^k z_i z_i^* \frac{(z - \frac{1}{z_i})(z - \frac{1}{z_i^*})}{(z - z_i)(z - z_i^*)} . \quad (A. 35)$$

Following the ideas of 'complementary' operator and the orthogonality in the z -domain, a set of discrete orthonormal exponentials may be written as

$$\begin{aligned} \psi_{2k-1}^s(z) &= b_{2k-1} \frac{z^2 - z}{(z - z_k)(z - z_k^*)} G_{k-1}^s(z) \\ \psi_{2k}^s(z) &= b_{2k} \frac{z^2 + z}{(z - z_k)(z - z_k^*)} G_{k-1}^s(z) \end{aligned} \quad (A. 36)$$

where

$$\begin{aligned} b_{2k-1} &= \left\{ \frac{T}{2} (1 - |z_k|^2) \left[1 + (z_k + z_k^*) + |z_k|^2 \right] \right\}^{1/2} \\ b_{2k} &= \left\{ \frac{T}{2} (1 - |z_k|^2) \left[1 - (z_k + z_k^*) + |z_k|^2 \right] \right\}^{1/2} . \end{aligned} \quad (A. 37)$$

The poles in the z -domain are related to the poles in the s -domain in such a way that

$$\begin{aligned} z_k &= e^{s_k T} = e^{(-\alpha_k + j\beta_k)T} \\ z_k^* &= e^{s_k^* T} = e^{(-\alpha_k - j\beta_k)T} \end{aligned} \quad (A.38)$$

We have assumed that the cardinal functions are used as the components for sampled data. As the sampling interval T approaches zero, our discrete orthonormal approach the ordinary continuous orthonormal exponential functions.